On the Distribution of Quadratic Forms

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Theorem 1 (Graybill 4.9)

$$x \sim N(\mu, V) \Rightarrow x'Ax \text{ is } \chi^{2'}(k, \tfrac{1}{2}\mu'A\mu) \text{ iff } AV \text{ idem}, \ r(A) = k.$$

Implicit in proof: V^{-1} existing.

We consider: V^{-1} non existent.

Lemma

$$E(x) \equiv E\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mu \equiv \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} I \\ K \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \mu_2 - K\mu_1 \end{bmatrix}$$

$$\text{iff } V \equiv \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I \\ K \end{bmatrix} V_{11}(I \quad K') \ .$$

Corollaries

1.1 $L' \equiv (I \quad K') \Rightarrow x = L(x_1 - \mu_1) + \mu$ iff $V = LV_{11}L'$.

1.2 $\mu_2 = K\mu_1$ (esp. $\mu = 0$) $\Rightarrow \mu = L\mu_1$, $x = Lx_1$, iff $V = LV_{11}L'$.

1.3 Smallest dimension for V_{11}: $r(V)$: V_{11}^{-1} exists.

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* Summary of paper BU-212-M, of like title, in the Mimeo Series, Biometrics Unit, Plant Breeding Department, Cornell University.

Theorem 2 (analogue to Theorem 1)

$$\left.\begin{array}{ll} x & \sim N(\mu, V) \\ \mu_2 & = L\mu_1 \\ V & = LV_{11}L' \\ r(V) & = r < n \\ V_{11}^{-1} & \text{existing} \end{array}\right\} \Rightarrow x'Ax \text{ is } \chi^{2'}(k, \tfrac{1}{2}\mu'A\mu) \ \underline{\text{iff}}\ L'AVAL = L'AL,\ k = tr(AV) \ .$$

Corollaries

2.1 $AVA = A \Rightarrow x'Ax$ is $\chi^{2'}[tr(AV), \tfrac{1}{2}\mu'A\mu]$.

2.2 AV idempotent $\Rightarrow x'Ax$ is $\chi^{2'}[tr(AV), \tfrac{1}{2}\mu'A\mu]$.

Both conditions sufficient, but not necessary.

Proofs 2.1 If $AVA = A$, then $L'AVAL = L'AL$.

2.2 If $AVAV = AV$, then $AVALV_{11}L' = ALV_{11}L'$,

so that $L'AVALV_{11}L'L = L'ALV_{11}L'L$,

and $(V_{11}L'L)^{-1} = (L'L)^{-1}V_{11}^{-1}$ exists.

A special case: Theorem 1

Graybill 4.9 (Theorem 1) is a special case: $L = I$, V^{-1} existing, implies

$L'AVAL = L'AL$ equivalent to $AVA = A$, $AVAV = AV$.

i.e. V^{-1} existing: necessary and sufficient condition is AV idempotent

V^{-1} non-existing: necessary and sufficient condition is

$L'AVAL = L'AL$.

Limitation to Theorem 2

Theorem applies only when $\mu = L\mu_1$.

This limits generality, but not importantly.

V usually singular because $x = Lx_1$, and so $\mu = L\mu_1$.

And for $\mu = 0$, $\mu = L\mu_1$ anyway.

From 1.1 $x = L(x_1 - \mu_1) + \mu$

And $(x_1 - \mu_1) \frown N(0, V_{11})$.

Hence $(x_1 - \mu_1)'L'AL(x_1 - \mu_1)$ is $\chi^{2'}(k, 0)$ iff $L'ALV_{11}$ idempotent

But this does not necessarily imply

$$[(x_1 - \mu_1)'L' + \mu']A[L(x_1 - \mu_1) + \mu] \text{ is } \chi^{2'} .$$

Applications

The necessary and sufficient condition of Theorem 2 involves L.

Sufficient condition of corollaries 2.1 and 2.2 do not.

But major interest is in conditions which lead to $x'Ax$ being $\chi^{2'}$ rather than consequences thereof.

Theorem 3 (Generalization of Corollary 2.1)

$$x \frown N(\mu, V),\ AVA = A \Rightarrow x'Ax \text{ is } \chi^{2'}[\mathrm{tr}(AV), \tfrac{1}{2}\mu'A\mu] .$$

(Limitation $\mu = L\mu_1$ not required.)

Conflict with Theorem of Rao's (1962)*

With $x \frown N(0, V)$, for $x'Ax$ to be χ^2_k:

2.1: $AVA = A$ is sufficient condition.

Rao: $AVA = A$ is necessary and sufficient condition.

Th. 2: $L'AVAL = L'AL$ is necessary and sufficient condition.

Example

$$V = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \quad : \quad x_3 = -(x_1 + x_2): \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

For $A = \frac{1}{16}\begin{bmatrix} 16 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}$ $x'Ax = \frac{1}{8}(8x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 5x_1x_3 + 3x_2x_3)$.

$$= \frac{1}{2}x_1^2 .$$

$\frown \chi^2_1$, as seen from V.

$$AVA = \frac{1}{64}\begin{bmatrix} 61 & 17 & 17 \\ 17 & 5 & 5 \\ 17 & 5 & 5 \end{bmatrix} \neq A$$

$$L'AVAL = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = L'AL .$$

* Rao, C. R. (1962) J.R.S.S.(B), 24, 152-158.